

# Analyzing Non-degenerate 2-Forms with Riemannian Metrics

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## Abstract.

We give a variational proof of the harmonicity of a symplectic form with respect to adapted riemannian metrics, show that a non-degenerate 2-form must be Kähler whenever parallel for some riemannian metric, regardless of adaptation, and discuss kählerness under curvature assumptions.

## Introduction

In the first part of this paper, we aim at a better understanding of the following folklore result (see [14, pp. 140-141] and references therein):

**Theorem 1** . *Let  $\sigma$  be a non-degenerate 2-form on a manifold  $M$  and  $g$ , a riemannian metric adapted to it. If  $\sigma$  is closed, it is co-closed for  $g$ .*

Let us specify at once a bit of terminology. We say a riemannian metric  $g$  is **adapted** to a non-degenerate 2-form  $\sigma$  on a  $2m$ -manifold  $M$ , if at each point  $x_o \in M$  there exists a chart of  $M$  in which  $g(x_o)$  and  $\sigma(x_o)$  read like the standard euclidean metric and symplectic form of  $\mathbb{R}^{2m}$ . If, moreover, this occurs for the first jets of  $\sigma$  (resp.  $\sigma$  and  $g$ ) and the corresponding standard structures of  $\mathbb{R}^{2m}$ , the couple  $(\sigma, g)$  is called an **almost-Kähler** (resp. a **Kähler**) structure; in particular,  $\sigma$  is then closed. Given  $\sigma$  non-degenerate, an adapted metric  $g$  can be constructed out of any riemannian metric on  $M$ , using Cartan's polar decomposition (cf. e.g. [13, lecture 2]). Let  $\mathcal{C}$  be the **conformal class** of a metric  $g$ , in other words, the set of metrics of the form  $e^f g$  with  $f$  a smooth real function on  $M$ ; we say that the class  $\mathcal{C}$  (resp. the metric  $g$ ) is adapted (resp. **conformally adapted**) to  $\sigma$ , if  $\mathcal{C}$  contains a metric adapted to  $\sigma$ . Last, we say that a metric is **co-closing** for an exterior form, if the form is co-closed with respect to the metric.

So theorem 1 implies that the symplectic form of an almost-Kähler structure is always *harmonic* with respect to its metric. As pointed out to us by V. Apostolov, theorem 1 is well-known in dimension 4, because then  $g$  adapted to  $\sigma$  implies  $*\sigma = \sigma$  (where  $*$

denotes the Hodge star operator, *cf.* note 1 below) hence  $\delta\sigma = - * d\sigma$  and  $\delta\sigma$  indeed vanishes when  $d\sigma = 0$ . In higher dimension, the proof given in [14, pp.140-141] goes by local calculations that yield the following formula (proving theorem 1):

$$(1) \quad i(\sigma)(d\sigma) = J^t(\delta\sigma) .$$

Here  $J$  is the almost-complex structure associated to  $\sigma$  and  $g$  ( $J^t$ , its transposed),  $\delta\sigma$  is the codifferential of  $\sigma$  with respect to  $g$ ,  $i(\sigma)$ , the interior product by  $\sigma$  *i.e.* the local  $g$ -adjoint of the exterior product map  $\alpha \mapsto \sigma \wedge \alpha$  (*cf.* *e.g.* [9, p.189-190]).

We found no direct variational proof of theorem 1 in the literature, a gap filled by the present note. The first step, of independent interest, is a new approach, riemannian as opposed to almost-complex, to the *adaptation* of metrics to a non-degenerate 2-form (section 1). It opens the way to an illuminating proof of theorem 1 and, not a surprise, to a stronger result in dimension 4 (section 2.1). In section 2.2, we briefly discuss the assumptions of theorem 1, namely the closedness of  $\sigma$  and the adaptation of  $g$ .

In the second part of the paper, we present proofs of a couple of old results in local almost-Kähler geometry, not so well-known actually, namely:

**Theorem 2 .** *Let  $\sigma$  be a 2-form on a connected manifold  $M$ . Assume  $\sigma$  is non-degenerate at one point and there exists a riemannian metric  $g$  on  $M$  for which  $\sigma$  is parallel. Then  $\sigma$  is Kähler (and so is  $g$ ).*

**Theorem 3 .** *Let  $(M, \sigma, g)$  be an almost-Kähler manifold. If the metric  $g$  is locally conformally flat, its scalar curvature must be non-positive, vanishing on  $M$  if and only if  $(M, \sigma, g)$  is Kähler.*

Theorem 2 is popular in case  $(M, \sigma, g)$  is almost-Kähler (*e.g.* [6]). It is stated in [9, p.251], with a sketchy proof though [9, pp. 211-213, 250-251] which we revisit in section 3 below (in particular, our proof of lemma 2 is new). A simpler result, assuming  $(M, g)$  *irreducible*, can be found in [14, pp.127-128].

Theorem 3 is proved, with detours, in [14, pp.194-197]. Here, we present a transparent proof of it (section 4) essentially based on the Weitzenböck formula for 2-forms combined with theorems 1 and 2. We also obtain stronger results in dimension 4.

Let us conclude with a bit of philosophy. Given a compact symplectic manifold which is *not* Kähler (a topological assumption), is there a best riemannian metric ? By theorem 1, asking for a co-closing metric is too weak a requirement, while by theorem 2, asking for parallelism of the 2-form is a too strong one. What can one find in-between ?

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## 1. Adaptation: the riemannian way

We start with a purely algebraic result (lemma 1) and use it on a manifold endowed with a non-degenerate 2-form, where it yields a conformal invariant. We will characterize conformally adapted metrics by minimizing the latter (theorem 4), then adapted ones by adding suitable constraints (theorem 5).

### 1.1 Positivity of a morphism and a related inequality

Let  $\sigma$  be a non-degenerate 2-form on a vector space  $E$  of dimension  $2m$ , endowed with a euclidean metric  $g$ . Let  $K$  denote the endomorphism of  $E$  defined by

$$(2) \quad \forall U \in E, g(KU, \cdot) = \sigma(U, \cdot).$$

It is, of course, skew-symmetric. Moreover, the morphism  $-K^2$  is symmetric positive-definite since  $g(-K^2U, U) = \sigma(-KU, U) = \sigma(U, KU) = g(KU, KU)$ . As such, it satisfies the inequality

$$(3) \quad \frac{1}{2m} \text{trace}(-K^2) \geq [\det(-K^2)]^{\frac{1}{2m}}.$$

Observe that  $\text{trace}(-K^2) = \text{trace}(\tilde{K}K) = 2|K|_g^2$ , where  $\tilde{K}$  is the  $g$ -transposed of  $K$  (here equal to  $-K$ ) and  $|\cdot|_g$  stands for the  $g$ -norm<sup>1</sup>. From (2) we have  $|K|_g^2 = |\sigma|_g^2$  and also

$$\det(K) = \frac{\det(\sigma)}{\det(g)}.$$

Since  $\det(-K^2) = [\det(K)]^2$ , we get

$$|\sigma|_g^2 \geq m \left| \frac{\det(\sigma)}{\det(g)} \right|^{\frac{1}{m}}$$

or else

$$|\sigma|_g^m \geq (m)^{\frac{m}{2}} \left| \frac{\det(\sigma)}{\det(g)} \right|^{\frac{1}{2}}.$$

Taking a Darboux co-basis for  $\sigma$  (in which  $\det(\sigma) = 1$ ) yields

$$\left| \frac{\det(\sigma)}{\det(g)} \right|^{\frac{1}{2}} \omega_g = \frac{1}{m!} \sigma^m$$

(where  $\omega_g$  stands for the volume form of  $g$ , oriented like  $\sigma^m$ ). We thus have proved:

**Lemma 1 .** *For any non-degenerate 2-form  $\sigma$  and euclidean metric  $g$  on a  $2m$ -vector space  $E$ , the following inequality holds:*

$$|\sigma|_g^m \omega_g \geq (m)^{\frac{m}{2}} \frac{1}{m!} \sigma^m.$$

Moreover, equality occurs if and only if the morphism  $K$  defined by (2) is a multiple of a complex structure on  $E$ .

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<sup>1</sup>the factor 2 is a standard convention due to skew-symmetry

The last part of the lemma follows from (3), where equality holds if and only if  $-K^2$  is a (positive) multiple of the identity.

## 1.2 Conformal adaptation

Let  $M$  be an oriented  $2m$ -manifold,  $\sigma$  a non-degenerate 2-form<sup>2</sup> and  $g$  a generic riemannian metric on  $M$ . For any open subset  $\Omega \subset M$  with compact closure, consider the following integral:

$$\mathcal{A}_m(g) = \int_{\Omega} |\sigma|_g^m \omega_g - (m)^{\frac{m}{2}} \frac{1}{m!} \sigma^m,$$

where  $\omega_g$ , the volume form of  $g$ , is oriented like  $\sigma^m$ . Henceforth, we denote by  $\mathcal{C}$  the conformal class of the metric  $g$ .

**Theorem 4** . *The functional  $\mathcal{A}_m$  is constant on  $\mathcal{C}$ , non-negative, vanishing if and only if  $g$  is conformally adapted to  $\sigma$  on  $\Omega$ . Moreover, the only critical value of  $\mathcal{A}_m$  is 0.*

According to this theorem, one may set  $A_m(\sigma, \mathcal{C}, \Omega) := \mathcal{A}_m(g)$  and view this functional as measuring how much  $\mathcal{C}$  departs from being adapted to  $\sigma$  on  $\Omega$ .

**Proof.** If  $g' = e^{2f}g$ , it is straightforward to check that

$$|\sigma|_{g'}^m \omega_{g'} = |\sigma|_g^m \omega_g,$$

proving the first statement. The second one entirely follows from lemma 1. It remains only to prove the third statement. To start with, the value 0 is indeed assumed by  $\mathcal{A}_m$  since there always exists on  $\Omega$  a riemannian metric adapted to  $\sigma$  [13, Lecture 2]. Last, assume that  $g$  is critical for  $\mathcal{A}_m$ . It means that for any symmetric covariant 2-tensor  $h$  on  $\Omega$ ,

$$\frac{d}{dt} \mathcal{A}_m(g + th)|_{t=0} = 0.$$

A routine calculation shows that this is equivalent to the following Euler's equation for  $g$  on  $\Omega$  (henceforth we use Einstein's convention):

$$(4) \quad g^{ab} \sigma_{ia} \sigma_{jb} = \frac{1}{m} |\sigma|_g^2 g_{ij}.$$

The latter really bears on  $\mathcal{C}$  since it is conformally invariant. Taking  $g$  in  $\mathcal{C}$  such that

$$|\sigma|_g^2 = m$$

and setting  $J_j^i := g^{ik} \sigma_{jk}$ , we infer at once from (4) multiplied by  $g^{ik}$  that  $J$  is an *almost-complex* structure on  $\Omega$ . In other words,  $\mathcal{C}$  must be adapted to  $\sigma$  on  $\Omega$ , hence  $\mathcal{A}_m(g) = 0$ . The theorem is proved.

**Remark 1** . Although not used below, let us record a pointwise (in fact vectorial) version of theorem 4, in terms of the *conformal adaptation function*  $a_m(g)$ , defined on  $(M, \sigma)$  by:

$$a_m(g) \frac{1}{m!} \sigma^m = |\sigma|_g^m \omega_g - (m)^{\frac{m}{2}} \frac{1}{m!} \sigma^m.$$

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<sup>2</sup>not necessarily closed

It readily goes as follows: for any  $x_0 \in M$ , the functional  $g \mapsto a_m(g)(x_0)$  is constant on  $\mathcal{C}$ , non-negative, vanishing if and only if  $g$  is conformally adapted to  $\sigma$  at  $x_0$ , with 0 as sole critical value.

### 1.3 Adaptation

Theorem 4 provides a criteria of *conformal* adaptation only. One can strengthen it and get a full test of adaptation by using Hölder's inequality, which we do now.

**Theorem 5** . Assume  $m > 1$ . A riemannian metric  $g$  in a conformal class  $\mathcal{C}$  is adapted to the non-degenerate 2-form  $\sigma$  on  $\Omega$  if and only if  $g$  satisfies  $A_m(\sigma, \mathcal{C}, \Omega) = 0$  and

$$Vol(\Omega, g) = \int_{\Omega} \frac{1}{m!} \sigma^m, \quad \int_{\Omega} |\sigma|_g \omega_g = \int_{\Omega} \frac{\sqrt{m}}{m!} \sigma^m,$$

where  $Vol(\Omega, g) := \int_{\Omega} \omega_g$ . Moreover, if  $m > 2$ , the last constraint can be replaced by

$$\int_{\Omega} |\sigma|_g^2 \omega_g = \int_{\Omega} \frac{1}{m-1!} \sigma^m.$$

**Proof.** The "only if" part of the theorem (like the one of the preceding remark) is known. Let us prove the "if" part and thus assume that the integral constraints hold on  $\Omega$ . By theorem 4, the vanishing of  $A_m(\sigma, \mathcal{C}, \Omega)$  implies that  $g$  is conformal to an adapted metric  $g_o$  on  $\Omega$ : set  $g = e^{2u}g_o$ . By Hölder's inequality, we have (recall  $m > 1$ ):

$$(5) \quad \int_{\Omega} |\sigma|_g \omega_g \leq \left( \int_{\Omega} |\sigma|_g^m \omega_g \right)^{\frac{1}{m}} [Vol(\Omega, g)]^{1-\frac{1}{m}}$$

with equality holding if and only if  $|\sigma|_g$  is constant (cf. e.g. [10, p.65-66]). Conformal invariance yields

$$\int_{\Omega} |\sigma|_g^m \omega_g \equiv \int_{\Omega} |\sigma|_o^m \omega_o$$

(where the subscript o refers to the conformal adapted metric  $g_o$ ), while the volume constraint of the theorem implies

$$Vol(\Omega, g) = Vol(\Omega, g_o).$$

Therefore the right-hand side of (5) equals  $\int_{\Omega} |\sigma|_o \omega_o \equiv \int_{\Omega} \frac{\sqrt{m}}{m!} \sigma^m$ . From the third constraint of the theorem, we see that equality holds in (5), so the conformal factor  $e^{2u}$  must be constant, equal to 1 due to the volume constraint. In other words,  $g \equiv g_o$  must indeed be adapted to  $\sigma$ . The last part of theorem 5 can be proved similarly, when  $m > 2$ , using instead of (5) the following Hölder's inequality:

$$\int_{\Omega} |\sigma|_g^2 \omega_g \leq \left( \int_{\Omega} |\sigma|_g^m \omega_g \right)^{\frac{2}{m}} [Vol(\Omega, g)]^{1-\frac{2}{m}}.$$

From the preceding proof, we infer also a *pointwise* characterization of adaptation (used in section 3.2 below), namely:

**Corollary 1** . *A riemannian metric  $g$  is adapted to a non-degenerate 2-form  $\sigma$  if and only if it satisfies:  $|\sigma|_g^2 = m$ ,  $\omega_g = \frac{1}{m!} \sigma^m$ .*

**Remark 2** . We thus have got only few equations for the set of adapted metrics on  $(\Omega, \sigma)$ , instead of the usual  $m(2m+1)$  pointwise equations  $K^2 = -I$  (for the metric  $g$ , in the notations of (2)). In return, our equations are not good for the purpose of a submanifold structure result. Indeed, the first equation of theorem 5 is singular, because 0 is *not* a regular value of  $A_m$  (it is an absolute minimum !); similarly, the two equations of corollary 1 both yield:  $\text{trace}_g(h) = 0$  when linearized along  $h$  at an *adapted* metric  $g$ . Nevertheless, those equations can be quite convenient for practical use.

## 2. Harmonicity

### 2.1 Variational proof of theorem 1

Theorem 4 yields the following statement, which implies theorem 1:

**Corollary 2** . *Let  $(M, \sigma_o)$  be a symplectic  $2m$ -manifold and  $\mathcal{C}$ , an adapted conformal class. Then  $\sigma_o$  must be co-closed with respect to any  $g \in \mathcal{C}$  such that  $|\sigma_o|_g$  is constant.*

**Proof.** Let  $\Omega$  be an open subset of  $M$  with compact closure and  $\alpha$ , a generic 1-form with compact support in  $\Omega$ . For  $t \in \mathbb{R}$  small enough, consider the symplectic form  $\sigma_t := \sigma_o + t d\alpha$ . From theorem 4 we know that

$$A_m(\sigma_t, \mathcal{C}, \Omega) \geq 0 = A_m(\sigma_o, \mathcal{C}, \Omega),$$

hence also that

$$\int_{\Omega} |\sigma_t|_g^m \omega_g \geq \int_{\Omega} |\sigma_o|_g^m \omega_g$$

since  $\sigma_t$  is cohomologous to  $\sigma_o$ . Now if we write

$$\frac{d}{dt} \left( \int_{\Omega} |\sigma_t|_g^m \omega_g \right)_{t=0} = 0$$

with a metric  $g \in \mathcal{C}$  such that  $|\sigma_o|_g$  is constant, we get equivalently

$$\frac{d}{dt} \left( \int_{\Omega} |\sigma_t|_g^2 \omega_g \right)_{t=0} = 0$$

(i.e. we are back to classical Hodge-de Rham theory). Since  $\alpha$  is compactly supported in  $\Omega$ , the latter yields (cf. e.g. [12, p.221])

$$\int_{\Omega} g(\alpha, \delta \sigma_o) \omega_g = 0;$$

taking for  $\alpha$  the 1-form  $\delta\sigma_o$  multiplied by a (non-negative) cut-off function and recalling  $\Omega$  is arbitrary, we conclude that  $\sigma_o$  must indeed be co-closed.

In case  $m = 2$  the preceding proof becomes trivial. Theorem 4 itself can then be complemented by:

**Proposition 1** . *Let  $M$  be a 4-manifold endowed with a conformal class  $\mathcal{C}$  and a non-degenerate 2-form  $\sigma$ . Then, for any open subset  $\Omega \subset M$  with compact closure, the condition  $A_2(\sigma, \mathcal{C}, \Omega) = 0$  holds if and only if  $*\sigma = \sigma$  on  $\Omega$ . If so, the closedness and co-closedness of  $\sigma$  on  $\Omega$  are equivalent.*

**Note 1** . For completeness and later use, let us recall basic facts about the Hodge star operator. For any metric  $g$  with volume form  $\omega_g$ , it can be defined by

$$\alpha \wedge *\beta = g(\alpha, \beta) \omega_g.$$

In dimension  $4k$  on  $2k$ -forms, it depends only on the conformal class  $\mathcal{C}$  of the metric  $g$ ; indeed then, the preceding right-hand side remains unchanged as  $g$  varies in  $\mathcal{C}$ . Furthermore, it is then an involution; so, if for a generic  $2k$ -form  $\alpha$  we set

$$\alpha^+ = \frac{1}{2}(\alpha + *\alpha), \quad \alpha^- = \frac{1}{2}(\alpha - *\alpha),$$

the latter satisfy:  $*\alpha^+ = \alpha^+$ ,  $*\alpha^- = -\alpha^-$ . In other words, the vector bundle of  $2k$ -forms splits into eigensubbundles (of equal dimension)  $\Lambda^{2k} = \Lambda^+ \oplus \Lambda^-$  respectively associated to the eigenvalues  $+1$  and  $-1$ . Last,  $\Lambda^+$  and  $\Lambda^-$  are orthogonal: indeed, for any  $\alpha \in \Lambda^+$  and  $\beta \in \Lambda^-$ ,

$$\alpha \wedge \beta = -\alpha \wedge *\beta = \beta \wedge *\alpha$$

so  $g(\alpha, \beta) = 0$  (and  $\alpha \wedge \beta = 0$ ).

**Proof of proposition 1.** If  $*\sigma = \sigma$  on  $\Omega$ , the last statement follows from the very definition of the co-derivative  $\delta$  (cf. e.g. [12, p.220]). Moreover, the form  $(\sigma \wedge *\sigma)$  is then both equal to  $|\sigma|_g^2 \omega_g$  (by note 1) and to  $\sigma^2$  on  $\Omega$ , hence indeed  $A_2(\sigma, \mathcal{C}, \Omega) = 0$ .

Conversely, assume  $A_2(\sigma, \mathcal{C}, \Omega) = 0$ . By theorem 4, there exists a metric  $g \in \mathcal{C}$  adapted to  $\sigma$  on  $\Omega$ . Classically it satisfies  $|\sigma|_g^2 = 2$ ,  $\omega_g = \frac{1}{2}\sigma^2$ ; therefore we have  $\sigma \wedge (\sigma - *\sigma) = 0$ . But, using note 1, we also have:

$$\begin{aligned} \sigma \wedge (\sigma - *\sigma) &= (\sigma^+ + \sigma^-) \wedge 2\sigma^- = 2\sigma^- \wedge \sigma^- \\ &= -2\sigma^- \wedge *\sigma^- = -2|\sigma^-|_g^2 \omega_g, \end{aligned}$$

hence  $\sigma^- = 0$  and  $*\sigma = \sigma$  on  $\Omega$ .

## 2.2 Discussion of theorem 1

How far are the assumptions of theorem 1 necessary to conclude that the non-degenerate 2-form  $\sigma$  is co-closed ?

The **closedness** of  $\sigma$  is definitely necessary in dimension 4, as explained in the introduction. It is no more the case in higher dimension. Indeed, in dimension 6, Gauduchon [4, pp.120-121] has constructed an example of a compact complex manifold, non-Kähler,

with a hermitian metric which is *co-closing* for its fundamental 2-form (such a structure is called semi-Kähler, and almost semi-Kähler in case the almost-complex structure is not integrable [14, p.192]). Taking products with 2-spheres yields examples in any dimension.

As regards the **adaptation** condition of theorem 1, we will now exhibit a co-closing metric which is not conformally adapted<sup>3</sup>. Take a compact almost-Kähler 4-manifold  $(M, \sigma_o, g)$  with  $b^- \neq 0$  (so, in particular,  $b_2(M) > 1$ ); for instance, a K3 surface [1, p.160].

**Proposition 2 .** *Let  $\alpha \in \Lambda^-$  be harmonic and  $t \in \mathbb{R}$  be small enough such that the 2-form  $\sigma_t := \sigma_o + t\alpha$  is non-degenerate. Then for  $t \neq 0$  the conformal class  $\mathcal{C}$  of the metric  $g$  is co-closing for the symplectic form  $\sigma_t$  but not adapted to it.*

**Proof.** Indeed, a straightforward computation yields

$$A_2(\sigma_t, \mathcal{C}, M) = 2t^2 \int_M |\alpha|_g^2 \omega_g > 0,$$

since  $g(\sigma_o, \alpha) = 0$  and  $\sigma_o \wedge \alpha = 0$  (cf. end of note 1). Therefore by theorem 4, for  $t \neq 0$  the class  $\mathcal{C}$  is not adapted to the symplectic form  $\sigma_t$ , although it is co-closing for it (cf. beginning of note 1).

### 3. Parallelism

#### 3.1 The parallel square root lemma

Let  $(M, g)$  be a riemannian  $n$ -manifold with Levi-Civita connection  $\nabla$ , and  $B$ , a field of symmetric endomorphisms on  $(M, g)$ . Assume  $B$  has positive eigenvalues and let  $A$  denote its positive square root. Let  $(e_1, \dots, e_n)$  be a local orthonormal frame field diagonalizing  $B$ ,  $(\lambda_1, \dots, \lambda_n)$  denote the corresponding (possibly non-distinct) eigenvalues of  $B$ , and  $(\theta^1, \dots, \theta^n)$ , the dual co-frame field. Locally we have (still with Einstein's convention):

$$B = \lambda_i e_i \otimes \theta^i, \quad A = \sqrt{\lambda_i} e_i \otimes \theta^i.$$

The aim of this section is to establish the following result.

**Lemma 2 .** *If  $B$  is parallel for  $\nabla$ , so is  $A$ .*

**Proof.** Fixing an arbitrary point  $x_o \in M$ , we shall prove that  $\nabla A(x_o) = 0$ . Let us take a normal chart  $(x^1, \dots, x^n)$  at  $x_o$  such that, setting  $\partial_j = \frac{\partial}{\partial x^j}$  and sticking to the preceding notations, the local matrix field  $(p_i^j)$  defined by:

$$e_i = p_i^j \partial_j$$

satisfies  $p_i^j(x_o) = \delta_i^j$ . Let  $(q_i^j)$  denote the inverse matrix field; it is such that

$$\theta^i = q_j^i dx^j.$$

<sup>3</sup>conformally adapted would be trivial in dimension 4



The expressions of  $B$  and  $A$  in the chart are:

$$\begin{aligned} B &= \lambda_i p_i^j q_k^i \partial_j \otimes dx^k \\ A &= \sqrt{\lambda_i} p_i^j q_k^i \partial_j \otimes dx^k. \end{aligned}$$

Covariantly differentiating, we get at  $x_o$ :

$$0 = \nabla_m B(x_o) = (\partial_m \lambda_i) \partial_i \otimes dx^i + \lambda_i [(\partial_m p_i^j) \partial_j \otimes dx^i + (\partial_m q_k^i) \partial_i \otimes dx^k]$$

and

$$\nabla_m A(x_o) = \frac{(\partial_m \lambda_i)}{2\sqrt{\lambda_i}} \partial_i \otimes dx^i + \sqrt{\lambda_i} [(\partial_m p_i^j) \partial_j \otimes dx^i + (\partial_m q_k^i) \partial_i \otimes dx^k].$$

There are two types of coefficients, namely those like the ones of  $\partial_1 \otimes dx^1$ :

$$0 = [\nabla_m B(x_o)]_1^1 = \partial_m \lambda_1 + \lambda_1 (\partial_m p_1^1 + \partial_m q_1^1),$$

$$[\nabla_m A(x_o)]_1^1 = \frac{\partial_m \lambda_1}{2\sqrt{\lambda_1}} + \sqrt{\lambda_1} (\partial_m p_1^1 + \partial_m q_1^1),$$

and those like the ones of  $\partial_1 \otimes dx^2$ :

$$0 = [\nabla_m B(x_o)]_2^1 = \lambda_2 \partial_m p_2^1 + \lambda_1 \partial_m q_2^1,$$

$$[\nabla_m A(x_o)]_2^1 = \sqrt{\lambda_2} \partial_m p_2^1 + \sqrt{\lambda_1} \partial_m q_2^1.$$

Let us pause for a remark: since  $g(e_i, e_j) = \delta_{ij}$ , we have

$$g(\nabla_m e_i, e_j) + g(e_i, \nabla_m e_j) = 0$$

hence at  $x_o$ , where

$$\nabla_m e_i(x_o) = (\partial_m p_i^j)(x_o) \partial_j,$$

we obtain

$$(\partial_m p_i^j + \partial_m p_j^i)(x_o) = 0;$$

similarly, we have

$$(\partial_m q_i^j + \partial_m q_j^i)(x_o) = 0.$$

Taking  $i = j = 1$ , we infer that:

$$[\nabla_m B(x_o)]_1^1 = 0 \Rightarrow \partial_m \lambda_1(x_o) = 0 \Rightarrow [\nabla_m A(x_o)]_1^1 = 0.$$

Moreover, taking  $i = 1, j = 2$ , and recording also the equality

$$0 = [\nabla_m B(x_o)]_1^2 = \lambda_1 \partial_m p_1^2 + \lambda_2 \partial_m q_1^2,$$

we get the following system at  $x_o$ :

$$\begin{aligned} \lambda_2 \partial_m p_2^1 + \lambda_1 \partial_m q_2^1 &= 0 \\ \lambda_1 \partial_m p_2^1 + \lambda_2 \partial_m q_2^1 &= 0. \end{aligned}$$

If  $\lambda_1 \neq \lambda_2$ , it implies that

$$\partial_m p_2^1(x_o) = \partial_m q_2^1(x_o) = 0.$$

If  $\lambda_1 = \lambda_2$ , then

$$(\partial_m p_2^1 + \partial_m q_2^1)(x_o) = 0.$$

In both cases, we obtain:

$$[\nabla_m A(x_o)]_2^1 = 0;$$

lemma 2 is proved.

### 3.2 Proof of theorem 2

With lemma 2 at hand, we can prove theorem 2 as follows.

The non-degeneracy and closedness of  $\sigma$  are standard. We only prove them for the sake of completeness.

**Non-degeneracy:** let  $y_o \in M$  such that  $\sigma(y_o)$  is non-degenerate. Since the manifold  $M$  is connected, for any fixed point  $y \in M$ , there exists a path  $\Gamma$  in  $M$  going from  $y$  to  $y_o$ . Moreover, the parallel transport  $\tau$  along  $\Gamma$  is an *isomorphism* from  $T_y M$  to  $T_{y_o} M$ , and  $\sigma$  parallel implies, for any couple  $(U, V)$  of  $T_y M$ :

$$\sigma(y)(U, V) = \sigma(y_o)(\tau U, \tau V).$$

Therefore the non-degeneracy of  $\sigma$  at  $y_o$  propagates to the point  $y$ . The latter being arbitrary,  $\sigma$  is indeed non-degenerate on all of  $M$ .

**Closedness:** since the Levi-Civita connection is torsionless,  $\sigma$  parallel implies  $d\sigma = 0$  according to the following formula, valid for any affine connection  $\nabla$  with torsion tensor  $T$ :

$$d\sigma(U, V, W) = \sum_{(U, V, W)} (\nabla_U \sigma)(V, W) + \sigma[T(U, V), W],$$

where  $\sum_{(U, V, W)}$  denotes *circular* summation on  $U, V, W$ .

**Kählerness:** this is the main point of theorem 2. In the presence of the metric  $g$  a skew-symmetric endomorphism field  $K$  is associated to the 2-form  $\sigma$  by  $g(KU, \cdot) = \sigma(U, \cdot)$ . When  $\sigma$  is  $g$ -parallel, so is  $K$ . Let us pause and assume as in [14, pp.127-128] that  $(M, g)$  is *irreducible*:  $K$  parallel then implies the existence of a positive constant  $c$  such that  $g(K\cdot, K\cdot) = cg$ , hence  $\sqrt{c}g$  is a Kähler metric with Kähler form  $\sigma$ . Back to the general case, the endomorphism field  $B := -K^2$ , which is symmetric positive-definite, is also  $g$ -parallel. By lemma 2, the positive square-root  $A$  of  $B$  must also be parallel, hence so is the field

$$J := KA^{-1}.$$

Classically, the latter is orthogonal and, by the uniqueness of Cartan's polar decomposition (cf. e.g. [13, lecture 2]), it satisfies  $J^2 = -I$  (almost-complex structure) and  $AJ = JA$ .

Now let us define a new metric  $g'$  by the formula

$$g'(U, V) := g(U, AV).$$

Since  $A$  is  $g$ -parallel, so is  $g'$ , hence  $g$  and  $g'$  have the *same* Levi-Civita connection. The almost-complex structure  $J$  is thus parallel for  $g'$ . It is also  $g'$ -orthogonal; indeed:

$$g'(JU, JV) = g(JU, AJV) = g(JU, JAV) = g(U, AV) = g'(U, V).$$

Noting the identity

$$\sigma(U, V) \equiv g'(JU, V),$$

we conclude that the 2-form  $\sigma$  and the metric  $g'$  define together a Kähler structure on  $M$ . Last, the formula

$$\sigma'(U, V) := g(JU, V)$$

defines a  $g$ -parallel (thus closed) non-degenerate 2-form on  $M$ ; therefore  $(\sigma', g)$  yields another Kähler structure. Theorem 2 is proved.

Dropping the non-degeneracy assumption on the 2-form  $\sigma$ , we now get:

**Corollary 3** . *Let  $(M, g)$  be a riemannian manifold with a parallel 2-form. Then the local de Rham decomposition of  $(M, g)$  has a Kähler factor of dimension equal to the rank of the 2-form. In case  $(M, g)$  is complete simply-connected, the same conclusion holds globally.*

**Proof.** Let  $\sigma$  be a parallel 2-form on  $(M, g)$ . It is closed, with constant rank  $r$  and we may assume  $r < \dim(M)$  without loss of generality (since if not, we are done by theorem 2). The orthogonal decomposition

$$TM = \text{Ker}(\sigma) \oplus \text{Ker}(\sigma)^\perp$$

is holonomy invariant; moreover, each factor is integrable and yields a totally geodesic foliation of  $M$  [5, p.180]. Given a generic point  $x_o \in M$ , let  $M'$  (resp.  $M''$ ) denote the leaf through  $x_o$  integral of  $\text{Ker}(\sigma)$  (resp. of  $\text{Ker}(\sigma)^\perp$ ) and  $g'$  (resp.  $g''$ ) the metric induced by  $g$  on it. Then  $(M, g)$  is locally isometric near  $x_o$  to the riemannian product  $(M' \times M'', g' \oplus g'')$  [5, p.182]; the same result holds globally provided  $(M, g)$  is complete simply-connected [5, p.187]. Now  $\sigma$  induces on  $M''$  a non-degenerate parallel 2-form, hence  $(M'', g'')$  is Kähler by theorem 2.

**Remark 3** . From the preceding proof, one can strengthen theorem 2 as follows in case  $(M, g)$  is irreducible (getting a statement stronger than in [14, p.128]):

**Theorem 6** . *If an irreducible riemannian manifold admits a non-zero parallel 2-form, then both the metric and the 2-form must be Kähler.*

## 4. Curvature conditions

### 4.1 A Weitzenböck formula

Throughout this section, the manifold  $M$  has dimension  $n > 2$ . In order to test when a riemannian metric  $g$  on  $M$  admits a parallel 2-form. and apply theorem 2 to conclude that

it is Kähler (provided the 2-form is non-degenerate), we need the Weitzenböck formula on 2-forms. Let us recall it [1, p.328]:

$$\nabla^* \nabla - (d\delta + \delta d) = \mathcal{R} - \frac{1}{2}r \oslash g.$$

Here  $\nabla$  denotes the Levi-Civita connection of  $g$ ,  $\nabla^*$  its formal  $L^2$  adjoint,  $r$  the Ricci operator,  $\oslash$  the Kulkarni-Nomizu product and  $\mathcal{R}$  the sectional curvature operator (viewed as a symmetric endomorphism of  $\Lambda^2 T^*M$ ). As well-known (cf. e.g. [2, p.48]), the latter admits the following *orthogonal* decomposition:

$$\mathcal{R} = \frac{s}{2n(n-1)}g \oslash g + \frac{1}{n-2}z \oslash g + \mathcal{W},$$

where  $s$  is the scalar curvature of  $g$ ,  $z$  its traceless Ricci tensor,  $\mathcal{W}$  its Weyl tensor. Moreover, for each 2-form  $\sigma$  on  $M$  the following identity holds:  $\frac{1}{2}(g \oslash g)(\sigma) \equiv \sigma$  (easy check). Therefore we have:

$$(6) \quad \nabla^* \nabla \sigma - (d\delta + \delta d)\sigma = -\frac{(n-2)s}{n(n-1)}\sigma - \frac{(n-4)}{2(n-2)}(z \oslash g)(\sigma) + \mathcal{W}(\sigma).$$

#### 4.2 Proof of theorem 3

In order to prove theorem 3 using (6), we need a couple of lemmas.

**Lemma 3** . *Let  $\sigma$  be a non-degenerate 2-form,  $\mathcal{C}$ , a riemannian conformal class adapted to  $\sigma$  and  $\zeta$ , a covariant symmetric 2-tensor, on a manifold  $M$ . Assume  $\zeta$  is traceless with respect to  $\mathcal{C}$ . Then, for any  $g \in \mathcal{C}$ , we have:  $g[\sigma, (\zeta \oslash g)\sigma] \equiv 0$ .*

**Proof.** From (4), there exists a positive function  $f$  on  $M$  such that:

$$g^{ab}\sigma_{ia}\sigma_{jb} = f g_{ij}.$$

Routine calculation then yields:

$$g[\sigma, (\zeta \oslash g)\sigma] \equiv 4f \operatorname{trace}_g(\zeta)$$

proving the lemma.

**Lemma 4** . *Let  $\sigma$  be a non-degenerate 2-form and  $g$ , a riemannian metric for which  $\sigma$  has constant norm. Then*

$$g(\sigma, \nabla^* \nabla \sigma) \equiv |\nabla \sigma|^2$$

(where  $|\cdot|$  stands for the  $g$ -norm).

**Proof.** One readily finds by direct calculation:

$$\frac{1}{2}\Delta(|\sigma|^2) = g(\sigma, \nabla^* \nabla \sigma) - |\nabla \sigma|^2$$

where  $\Delta$  denotes the (positive) laplacian of  $g$ . Whenever  $|\sigma|^2$  is *constant*, the left-hand side vanishes and the lemma follows.

We are now in position to prove theorem 3. Let  $(M, \sigma, g)$  be a conformally flat almost-Kähler manifold of dimension  $n = 2m$ . By theorem 1, the form  $\sigma$  is harmonic, so (6) yields (with  $\mathcal{W} = 0$ ):

$$\nabla^* \nabla \sigma = -\frac{(m-1)s}{m(2m-1)} \sigma - \frac{(m-2)}{2(m-1)} (z \otimes g)(\sigma).$$

Taking the scalar product with  $\sigma$  and using lemma 3 with  $\zeta = z$  and lemma 4 (which holds since  $|\sigma|^2 = m$ ), we obtain:

$$|\nabla \sigma|^2 = -\frac{(m-1)s}{m(2m-1)} |\sigma|^2.$$

Therefore  $s$  must be non-positive, and  $s \equiv 0$  is equivalent to kählerness by theorem 2.

**Remark 4** . Using Bochner's method for (6) combined with theorem 2 and corollary 3, we readily get in full generality (setting  $dV$  for the canonical Lebesgue measure):

**Theorem 7** . *Let  $(M, g)$  be a compact riemannian  $n$ -manifold,  $n > 2$ , and  $\sigma$ , a harmonic form on it. Assume either  $n = 4$  or  $g$  is Einstein. Then*

$$\int_M g[\sigma, \mathcal{W}(\sigma)] dV \geq \frac{(n-2)}{n(n-1)} \int_M s |\sigma|^2 dV.$$

*Equality holds if and only if  $\sigma$  is parallel. If so, either  $(M, g)$  is Kähler or a Kähler factor (of dimension the rank of  $\sigma$ ) splits off from its universal covering manifold endowed with the pulled-back metric.*

### 4.3 Further results in dimension 4

From (6), we see that harmonic 2-forms in dimension  $n = 4$  satisfy

$$(7) \quad \nabla^* \nabla \sigma = \mathcal{W}(\sigma) - \frac{s}{6} \sigma.$$

Furthermore, when  $n = 4$  the Weyl tensor commutes with the Hodge star operator [11]; it thus splits according to  $\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-$ , with  $\mathcal{W}^\pm$  a symmetric morphism of  $\Lambda^\pm$  (cf. note 1).

**Theorem 8** . *Let  $(M, \sigma, g)$  be almost-Kähler 4-dimensional. If  $\mathcal{W}^+ = 0$  then  $s \leq 0$ , and  $s \equiv 0$  if and only if the manifold is Kähler.*

The proof is similar to that of theorem 3; the weaker assumption  $\mathcal{W}^+ = 0$  suffices here because  $\sigma \in \Lambda^+$  (cf. proposition 1).

In case  $M$  is *compact* orientable, we can prove a stronger result, namely:

**Theorem 9** . Let  $M$  be a compact orientable 4-manifold endowed with a riemannian conformal class  $\mathcal{C}$ . Assume either  $b^+ > 0$  with  $\mathcal{W}^+ = 0$ , or  $b^- > 0$  with  $\mathcal{W}^- = 0$ . Then the Yamabe invariant  $\mu$  of  $\mathcal{C}$  is non-positive, and if  $\mu = 0$ , the class  $\mathcal{C}$  contains a Kähler metric.

For completeness, let us recall that the Yamabe invariant  $\mu$  of the conformal class  $\mathcal{C}$  is the *infimum*, for  $g \in \mathcal{C}$ , of the Hilbert action functional:

$$\mathbf{S}(g) = [\text{Vol}(M, g)]^{\frac{2}{n}-1} \int_M s \omega_g.$$

From theorem 9 we deduce at once the:

**Corollary 4** . Let  $(M, \mathcal{C})$  be a compact orientable 4-manifold endowed with a riemannian conformal class. If  $\mathcal{W} = 0$  and  $\mu > 0$ , then  $b_2 = 0$ .

**Remark 5** . Since  $n = 4$ , no assumption need be made on the Ricci tensor  $r$  (see (6)). In higher dimension, the vanishing of  $b_2$  (even that of  $b_i$  for  $i < n$ ) is known when  $\mathcal{W} = 0$  and  $r$  is positive definite, in particular in case of constant positive curvature [8, p.6].

**Proof of theorem 9.** Assume  $b^\pm > 0$  and  $\mathcal{W}^\pm = 0$ . Let  $\sigma \in \Lambda^\pm$  be harmonic, with  $\sigma \neq 0$ . It is non-degenerate at one point; indeed, otherwise, for any  $g \in \mathcal{C}$ :

$$0 = \int_M \sigma^2 \equiv \pm \int_M |\sigma|^2 \omega_g$$

which contradicts  $\sigma \neq 0$ . Applying Bochner's method to (7) with  $g \in \mathcal{C}$  such that  $|\sigma|^2$  is constant, we obtain, using the half-conformal flatness:

$$(8) \quad \int_M |\nabla \sigma|^2 \omega_g = -\frac{1}{6} |\sigma|^2 \int_M s \omega_g.$$

So  $\int_M s \omega_g$  (the so-called total scalar curvature of  $g$ ) must be non-positive. It implies  $\mu \leq 0$  as claimed. Last, if  $\mu = 0$ , any metric in  $\mathcal{C}$  must have its total scalar curvature *non-negative*. In particular, the preceding one thus satisfies  $\int_M s \omega_g = 0$ . From (8) and theorem 2, it must be Kähler. Theorem 9 is proved.

**Remark 6** . Let  $\mathcal{C}$  be the conformal class of the metric  $g$  in theorem 8, and  $\mu$ , the Yamabe invariant of  $\mathcal{C}$ . If  $g$  is Kähler, then of course  $\mu = 0$ . Let us prove it for completeness: since  $s = 0$ , the scalar curvature  $s'$  of a generic metric  $g' \in \mathcal{C}$  written as  $g' = u^2 g$  (with  $u$  a positive function) is given by (e.g. [7, p.38]):

$$s' = 6u^{-3} \Delta u,$$

where  $\Delta = \delta d$  stands for the laplacian of  $g$ . So the Hilbert functional  $\mathbf{S}(g')$  is *non-negative*, equal to

$$\mathbf{S}(g') = 6 \left( \int_M u^4 \omega_g \right)^{1/2} \int_M |du|^2 \omega_g.$$

Therefore  $\mu = 0$  as claimed.

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